

A COMMENT OF THE COMBINATORICS OF THE VERTEX OPERATOR $\Gamma_{(t|X)}$

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ABSTRACT. The Jacobi–Trudi identity associates a symmetric function to any integer sequence. Let $\Gamma_{(t|X)}$ be the vertex operator defined by $\Gamma_{(t|X)}s_\alpha = \sum_{n \in \mathbb{Z}} s_{(n,\alpha)}[X]t^n$. We provide a combinatorial proof for the identity $\Gamma_{(t|X)}s_\alpha = \sigma[tX]s_\alpha[x - 1/t]$ due to Thibon et al., [12, 14]. We include an overview of all the combinatorial ideas behind this beautiful identity, including a combinatorial description for the expansion of $s_{(n,\alpha)}[X]$ in the Schur basis, for any integer value of n .

1. INTRODUCTION

This short note concerns a powerful operator on symmetric functions, the vertex operator $\Gamma_{(t|X)}$. Let α be a partition, and let s_α be the corresponding Schur function, then $\Gamma_{(t|X)}$ is the generating function for Schur functions indexed by those integer sequences obtained from α by prepending an integer part:

$$(1.1) \quad \Gamma_{(t|X)}s_\alpha = \sum_{n \in \mathbb{Z}} s_{(n,\alpha)}[X]t^n$$

This vertex operator is a classical tool in the theory of symmetric functions. It has been used by Thibon and his collaborators for the study of various phenomena of stability [6, 8, 14, 11, 12]. Previously, it had been used in mathematical physics, Jing [7]. Likewise, $\Gamma_{(t|X)}$ is the generating series for Bernstein’s creation operators introduced in [15]. Recently, $\Gamma_{(t|X)}$ has played a central role in our work on operators on symmetric functions [2], and on the growth of the Kronecker coefficients [4, 5].

The aim of this note is to provide a combinatorial proof for the identity

$$(1.2) \quad \Gamma_{(t|X)}s_\alpha = \sigma[tX]s_\alpha[x - 1/t]$$

by means of a sign reversing involution. While concise algebraic proof can be found in [12, 14], readers with a more combinatorial taste may find this proof interesting.

We take the opportunity to present a compilation of results (some of them very well-known, other less so) needed to interpret all functions appearing in Eq. 1.2 combinatorially. As usual, we follow the notation of Macdonald’s book [10], unless it is explicitly stated. A point of depart will be that we draw our partitions the French way. Lascoux’s approach to symmetric functions permeates this note, [9].

The famous Jacobi–Trudi identity express Schur function in the complete homogeneous basis $s_{(\alpha_1, \alpha_2, \dots, \alpha_l)} = \det(h_{\alpha_i + j - i})_{i,j=1 \dots l}$ where, as usual, the h_k ’s are the complete homogeneous symmetric functions, $h_0 = 1$ and $h_k = 0$ when $k < 0$.

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While it is a well-known theorem that the previous identity holds for any partition (see for example the beautiful combinatorial proof of Gessel and Viennot using lattice paths), the Jacobi-Trudi determinant makes perfectly good sense for any integer sequence α . This allows us to make sense of the definition of the vertex operator $\Gamma_{(t|X)}$, that deals with functions s_α whose first part is out-of-order or even nonpositive. It turns out that for any integer sequence, the Jacobi-Trudi determinant s_α is equal to either zero, or to \pm a Schur function s_λ , with λ a partition. In this situation, we say that the partition λ is the *rectification* of α .

The process of rectifying a shape coming from an integer sequence is better described using an example. The identity $s_{5,3,2,7} = s_{5,4,4,3}$ can be shown to hold writing the corresponding Jacobi-Trudi determinant, and performing row exchanges:

$$\begin{vmatrix} h_5 & h_6 & h_7 & h_8 \\ h_2 & h_3 & h_4 & h_5 \\ 1 & h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 & h_7 \end{vmatrix} = (-) \begin{vmatrix} h_5 & h_6 & h_7 & h_8 \\ h_2 & h_3 & h_4 & h_5 \\ h_4 & h_5 & h_6 & h_7 \\ 1 & h_1 & h_2 & h_3 \end{vmatrix} = (+) \begin{vmatrix} h_5 & h_6 & h_7 & h_8 \\ h_4 & h_5 & h_6 & h_7 \\ h_2 & h_3 & h_4 & h_5 \\ 1 & h_1 & h_2 & h_3 \end{vmatrix}$$

Note that the weight of the rectified partition coincides with weight of the original integer sequence (i.e., the sum of the parts of the integer partition). This process is sometimes known as Littlewood's modification rule.

It is well-known that we can perform the row exchanges on the Jacobi-Trudi determinants directly on the tableaux indexing these determinant “by letting gravity act”. Draw the try diagrams corresponding to the Jacobi-Trudi identities appearing in our running example:

$$\begin{array}{c} \begin{array}{cccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & & & & \\ \blacksquare & \blacksquare & \blacksquare & & & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\ (5, 3, 2, 7) \end{array} = (-) \begin{array}{c} \begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\ (5, 3, 6, 3) \end{array} = (+) \begin{array}{c} \begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\ (5, 5, 4, 3) \end{array}$$

The exchange of the last two rows in the Jacobi-Trudi determinant can be achieved by fixing the last cell on the third row, and the one North-East to it (here, these two cells are drawn in grey), and letting the cell to the right of the grey cell in the top row fall into the third row.

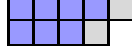
$$\begin{array}{c} (+) \begin{array}{cccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & & & & \\ \blacksquare & \blacksquare & \blacksquare & & & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\ (5, 3, 2, 7) \end{array} = (+) \begin{array}{c} \begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\ (5, 3, 6, 3) \end{array} = (-) \begin{array}{c} \begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\ (5, 3, 6, 3) \end{array}$$

The result is still not a partition. We repeat the previous procedure with the next two rows:

$$\begin{array}{c} (-) \begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\ (5, 3, 6, 3) \end{array} = (-) \begin{array}{c} \begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\ (5, 5, 4, 3) \end{array} = (+) \begin{array}{c} \begin{array}{cccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\ (5, 5, 4, 3) \end{array}$$

Note that, if at some point a sequence can not be rectified using this procedure, then the corresponding Jacobi-Trudi determinant is zero. This situation is illustrated in the next picture, where there is not cell to move from the first to

the second row. Remark that this situation occurs because in the corresponding Jacobi–Trudi determinant there is a repeated row.



$$s_{(4,5)} = 0$$

1.1. The rectification of $s_{(n,\alpha)}$. We need an explicit description of those partitions that can occur as the result of rectifying an integer sequences, where the first part is the only one allowed to be out-of-order or nonpositive.

Lemma 1.1. *Let α be a partition, then after rectification,*

$$\sum_{n \in \mathbb{Z}} s_{(n,\alpha)}[X] = \sum_{\mu} (-1)^{ht(\mu)} s_{\mu}$$

where we sum over the set of partitions μ obtained from α by

- (1) Fixing some $k \geq 0$,
- (2) Adding to α a cell to the first k columns, (forming a horizontal strip),
- (3) Removing a cell from α from each row $\alpha_i > k$, (forming a vertical strip),

and where $ht(\mu)$ be the number of cells in this vertical strip .

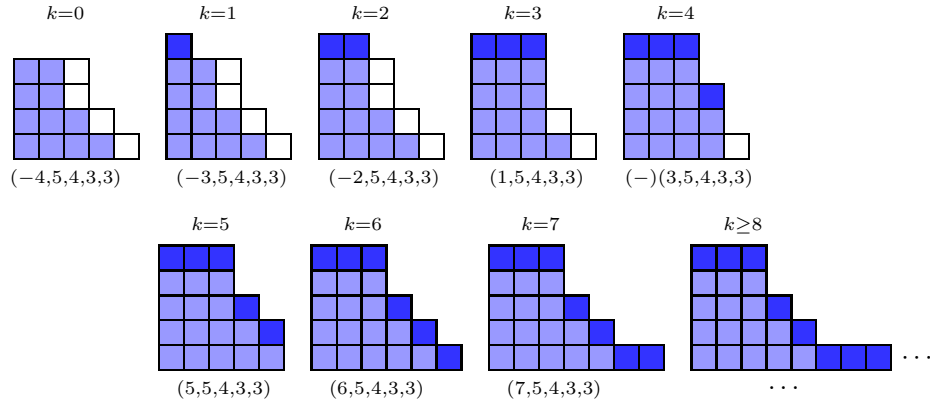
Before providing a proof for the result, let us look at an example.

Example 1.2 (Rectifications of $\alpha = (n, 5, 4, 3, 3)$, with $n \in \mathbb{Z}$). It is easy to check that if $n < -4$ the Jacobi–Trudi determinant $s_{(n,5,4,3,3)} = 0$, as the first row of this determinant is identically zero.

We illustrate all possible partitions μ obtained by rectifying $\alpha = (k, 5, 4, 3, 3)$, for $k \geq -4$ in the following Figure.

To make explicit the comparison of α with its rectified partition μ , we draw in white those cells of α that are not in μ (forming a vertical strip), and in dark blue boxes those cells added that where initially not in α but that are in μ (forming a horizontal strip). That is to say, the original integer sequence α is the union of the white and the pale blue cells, and its rectification λ is the union of both shades of blue.

Finally, we write the integer sequence α below the diagram of rectified partition it corresponds, and write the parameter k (the index of Lemma 1.1) on top of it.



Note that boxes are deleted from α precisely when $n < \alpha_1$. Also, sequence $(3, 5, 4, 3, 3)$ is the only one that rectifies to a partition negative sign.

Definition 1.3 (α -removable). Let α be an integer sequence where only the last part is allowed to be out-of-order or nonnegative, and let μ be its rectification. Draw α and μ as before. Then, a cell of $\alpha \cap \mu$ will be called a α -removable if neither it has a blue cell on top, nor a white cell to its right.

Remark 1.4. The set of partitions appearing in Lemma 1.1 consists of those partitions that can be obtained from α by adding a horizontal strip, and deleting a vertical strip, and that leave no α -removable corners.

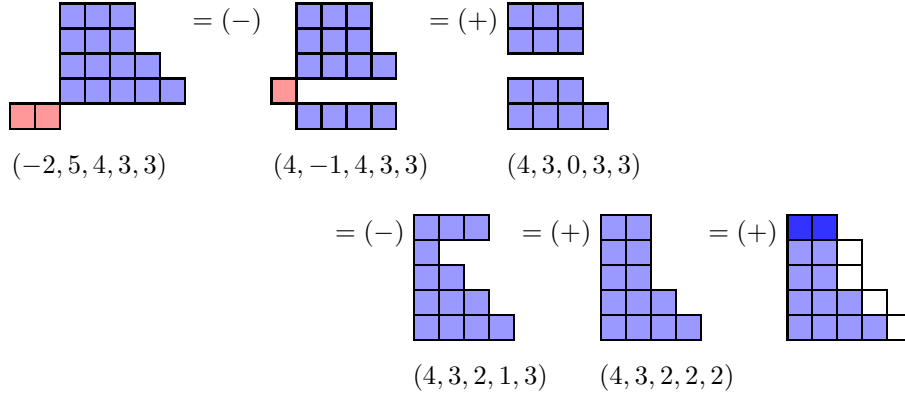
Proof. Suppose that α rectifies to a partition μ .

After exchanging rows r_k and r_{k+1} , the size of r_{k+1} (now at the r^{th} -row) decreases by 1. This part of the procedure is responsible for the appearance of the vertical strip. On the other hand, the size of r_k (now at position $r + 1$) increases by 1.

Originally, there only the first element of the integer sequence was out-of-order. This is the part that keeps moving up in the rectification process. Eventually, it arrives to a position where the sequence is weakly increasing and positive (otherwise the procedure would have failed). Looking at the diagram, this has the effect of adding a horizontal strip of α . The size of this horizontal strip is the parameter k appearing in the lemma.

Finally, notice that each time that this part goes up, the part that it is exchanged with loses a cell. As a result, there will be no α -removable cells in $\alpha \cap \mu$. \square

Example 1.5. The rectification of $s_{(-2,5,4,3,3)}$:



Where we use the same conventions for colors as in the previous example, and additionally, denoting negative parts with pale red.

It may be interesting to compare those partitions appearing in the previous lemma, with those of Theorem 1.1 of [3], where a formula is provided that allows to compute the reduced Kronecker coefficients from the Kronecker coefficients.

1.2. Operations of alphabets. We discuss some basic facts regarding operations on alphabets. First, notice that it is customary to write a morphism of algebras A from Sym to some commutative algebra \mathcal{R} as $f \mapsto f[A]$ (rather than $f \mapsto A(f)$), and consider it as a “specialization at the virtual alphabet A .”

Since the power sum symmetric functions p_k ($k \geq 1$) generate Sym and are algebraically independent, the map $A \mapsto (p_1[A], p_2[A], \dots)$ is a bijection from the set of

all morphisms of algebras from Sym to \mathcal{R} to the set of infinite sequences of elements from \mathcal{R} . This set of sequences is endowed with its operations of component-wise sum and product, and multiplication by a scalar. The bijection is used to lift these operations to the set of morphisms from Sym to \mathcal{R} .

In this way, expressions like $f[A+B]$ and $f[AB]$, where f is a symmetric function and A and B are two “virtual alphabets,” are defined. Note that, by definition, for any power sum p_k ($k \geq 1$), virtual alphabets A and B , and scalar z ,

$$p_k[A+B] = p_k[A] + p_k[B], \quad p_k[AB] = p_k[A] \cdot p_k[B], \quad p_k[zA] = z p_k[A].$$

In particular $p_1[A] = A$, $p_1[A+B] = A+B$, and so on.

The morphism $f \mapsto f^\perp = D_f$ associates to f the adjoint of the operator “multiplication by f ”, see [9, 10]. In particular, this morphism defines a somehow mysterious alphabet $X^\perp = p_1^\perp[X]$ that makes the following identity hold $f^\perp[X] = f[X^\perp]$. Later on, a combinatorial interpretation for both, $p_k^\perp[X]$ and X^\perp , will be provided.

1.3. The series σ . Let $\sigma = \sum_{n=0}^{\infty} h_n$ be the generating function for the complete homogeneous symmetric functions. From the definition of $h_k[A]$ as the sum of all square-free monomials of degree k in alphabet A , the following two well-known identities readily follow:

$$\sigma[X] = \prod_{x \in X} \frac{1}{1-x}$$

$$\sigma[X+A] = \sigma[X]\sigma[A].$$

On the other hand, the Robinson–Schensted–Knuth correspondence provides a combinatorial proof of the identity $\sigma[AB] = \sum_{\lambda} s_{\lambda}[A]s_{\lambda}[B]$, known as “Cauchy’s kernel”. This is because $\sigma[AB]$ is the generating functions for biwords, and because to each such biword the RSK algorithm associates a pair of semistandard tableaux of the same shape (bijectively). The fact that Schur functions are the generating functions for semistandard tableaux completes the argument.

1.4. Schur functions of a negative alphabet. We need a combinatorial interpretation for Schur functions evaluated in a difference of two alphabets. Again, this is carefully done in Lascoux’s book, working with operations on alphabets and Jacobi–Trudi determinants. Another interesting approach is given by a formula of Sergeev–Pragacz that expresses a Schur function evaluated at the difference of two alphabet in terms of Vandermonde determinants, a combinatorial proof for this formula can be found in [1]. Here we will use its combinatorial definition in terms of fillings of tableaux, that we present for the sake of completeness.

From the identity $\sigma[X+A] = \sigma[X]\sigma[A]$, setting $A = -X$ we conclude that $1 = \sigma[0] = \sigma[X]\sigma[-X]$. Therefore

$$\sigma[-X] = \frac{1}{\sigma[X]} = \prod_{x \in X} (1-x) = \prod_{x \in X} (1+(-x)) = \sum (-1)^k e_k[X]$$

Therefore, applying the the involution $X \mapsto -X$ to σ , and we conclude that

$$(1.3) \quad h_k[X] \mapsto (-1)^k e_k[X]$$

These two symmetric functions are, respectively, the generating functions for k -multisets/sets on alphabet X , the most classical instance of combinatorial reciprocity, [13].

Let X and Y be two alphabets. We are interested in the new alphabet $X - Y$, where we call the elements of X “positive letters”, whereas the ones of $-Y$ are the “negative letters”. A total order on $X - Y$ is usually defined by saying that negative letters are smaller than the positive ones, and by breaking ties using the absolute value of the entries.

Definition 1.6 (signed tableau). Let $X - Y$ be an alphabet defined as the difference of two positive alphabets.

A signed tableau T of shape λ is a filling of the diagram of λ with entries in $X - Y$ such that the entries are weakly increasing in the rows and columns, and

- (1) Positive letters are strictly increasing on the columns.
- (2) Negative letters are strictly increasing on the rows.

The weight of a letter depends on its sign. Let a be a positive letter, then a has weight x_a , and the corresponding negative letter $-a$ has weight $-y_a$. The weight of a signed tableaux is defined as usual.

We denote by T_+ be the subtableau of T positive letters, and by T_- the subtableau of the negative ones.

Example 1.7. A signed tableau T of shape $(6, 5, 4, 2, 1)$:

5					
4	4				
-1	-3	3	3		
-1	-3	1	2	3	
-1	-2	-3	1	2	2

where the subtableau T_+ is shaded blue, whereas T_- is shaded red. The signed tableaux T has weight:

$$x_1^2 x_2^3 x_3^3 x_4^2 x_5 (-y_1)^3 (-y_2) (-y_3)^3 = -x_1^2 x_2^3 x_3^3 x_4^2 x_5 y_1^3 y_2 y_3^3$$

Theorem 1.8 (A Schur function evaluated in the difference of two alphabets). *Let λ be a partition, and let X and Y be two alphabets, then*

$$s_\lambda[X - Y] = \sum_{T=T_- \cup T_+} x^{T_+} y^{T_-}$$

where we are summing over all possible signed tableaux $T = T_- \cup T_+$ of shape λ .

Proof. Theorem 1.8 follows from the identity $s_\lambda[X + Y] = \sum_{\mu \in \lambda} s_\mu[X] s_{\lambda/\mu}[Y]$ (immediate from the combinatorial definition of Schur function in terms of fillings of tableaux), combined with Eq. 1.3, and the dual Jacobi-Trudi identity (the determinant expression for a Schur function on the elementary symmetric basis). \square

If the negative alphabet $-Y = \{-t\}$ has just one letter, then condition (2) implies that T_- defines a vertical strip. Moreover, since $s_\lambda[X - Y]$ is a symmetric function, we can use any total order for the letters of $X - Y$ in its combinatorial definition. Setting negative letters to be bigger than the positive ones makes it transparent that $s_\lambda[X - Y]$ is the sum of all Schur functions $t^\ell S_\alpha[X]$, where α can be obtained from λ by removing a vertical strip of size ℓ . We have obtained the following corollary:

Corollary 1.9. *Let α be a partition, then*

$$s_\alpha[X - 1/t] = \sum_{k \geq 0} \sum_{\substack{\lambda \text{ partition} \\ \alpha \setminus \lambda \text{ } k\text{-vertical strip}}} (-1/t)^k s_\lambda[X]$$

From the combinatorial definition of a Schur function evaluated in a negative alphabet Cauchy's dual identity follows:

$$\sigma[-AB] = \sum_{\lambda} (-1)^{|\lambda|} s_{\lambda'}[A] s_{\lambda}[B]$$

1.5. The alphabet X^\perp . The object of this section is to provide a combinatorial interpretation for the specialization $p_n^\perp[X] = p_n[X^\perp]$. We recall the following classical result, [9, p. 25].

Lemma 1.10. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a sequence of positive integers*

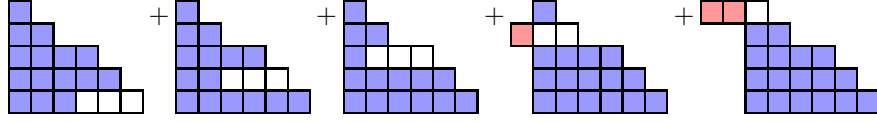
$$p_i^\perp s_\alpha = \sum_{k=1}^n s_{(\alpha_1, \dots, \alpha_k - i, \dots, \alpha_n)}$$

Proof. It is immediate that $p_i^\perp h_k = h_{k-i}$, as the coefficients of h_n in the power sum basis are one, for all p_λ , with $\lambda \vdash n$.

On the other hand, if we develop the Jacobi–Trudi determinant, and then apply the product rule, the result follows easily by regrouping together all the summand where the same h_k has been derived. \square

Example 1.11 (The effect of the perp operator).

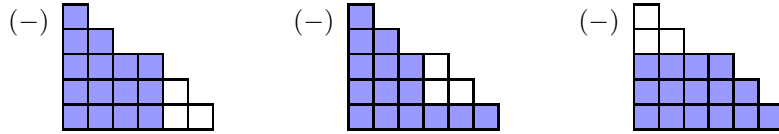
$$p_3^\perp s_{(6,5,4,2,1)} = s_{(3,5,4,2,1)} + s_{(6,2,4,2,1)} + s_{(6,5,1,2,1)} + s_{(6,5,4,-1,1)} + s_{(6,5,4,2,-2)}$$



If we rectify the resulting sequences we then obtain:

$$p_3^\perp s_{(6,5,4,2,1)} = -s_{(4,4,4,2,1)} - s_{(6,3,3,2,1)} + s_{(6,5,4)} - s_{(6,5,4,2,-2)}$$

The partitions indexing nonzero Schur functions are precisely



because sumands 3 and 5 rectify to zero.

As the reader probably suspects, if we rectify the partitions appearing in Lemma 1.10 we obtain the classical Murnaghan–Nakayama rule. Details can be found in [9]. On the other hand, the Murnaghan–Nakayama rule can be derived just as easily from the previous lemma using the combinatorial interpretation just provided.

The definition of specialization of a symmetric function at a virtual alphabet A needs the understanding of $p_n[A]$, for all $n \geq 0$. From a combinatorial point of view, the effect of the p_i^\perp operator on a Jacobi–Trudi determinant is to subtract n cells from each of the rows of α , one at a time. On the other hand, if we fix an alphabet X , then the identity $p_n^\perp[X] = p_n[X^\perp]$ provides us with a plausible combinatorial

interpretation for $(x_k^n)^\perp$: we say that $(x_k^n)^\perp$ acts on $s_\alpha[X]$ by deleting n cells from the k^{th} row of α . This makes sense as $s_\alpha[X]$ is nonzero only if $|X| \geq \ell(\lambda)$ so we will be subtracting k cells for each of the rows, successively.

1.6. The vertex operator $\Gamma_{(t|X)}$. Recall that the vertex operator $\Gamma_{(t|X)}$ is defined by $s_\alpha = \sum_{n \in \mathbb{Z}} s_{(n,\alpha)}[X]t^n$. We present a sign-reversing involution that shows that the following identity holds.

Lemma 1.12. *Let α be a partition.*

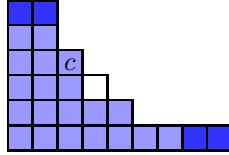
$$(1.4) \quad \Gamma_{(t|X)} s_\alpha = \sigma[tX] s_\alpha [x - 1/t]$$

Proof. First, we apply Lemma 1.9 followed with Pieri's rule to the expression $\sigma[tX] s_\alpha [x - 1/t]$ (which is valid since the sequences involved are always partitions). The partitions indexing a nonzero Schur function that appear as summands are precisely those that can be obtained by first deleting a vertical strip, and then adding a horizontal one. The sign of the corresponding Schur function is given by $(-1)^{|\text{vertical strip}|}$.

Since this is a signed sum, there are plenty of potential cancelations in this sum. Let us look at them closely. Let μ be a partition that indexes one of the summands of $\sigma[tX] s_\alpha [x - 1/t]$ (before carrying out any cancellation). We will refer to such a summand directly by the indexing partition μ .

We classify the resulting partitions into two classes. Those partitions μ that have a α -removable corner, and those that have not.

Suppose that there are α -removable corner in μ , and denote c be the leftmost one.



The summand indexed by μ appears indexing twice in $\sigma[tX] s_\alpha [x - 1/t]$: Once where corner c of μ was not removed as part of the vertical strip. The other one, when c was indeed removed as part of the vertical strip, but then added again by the horizontal strip. Our involution will pair this two tableaux.

On the other hand, if there is no α -removable corner in μ , then it will be paired to itself.

Notice that if two different tableaux are paired together, then they have the same weight (as the factors t and $1/t$ will cancel each other), but their signs will differ. Therefore the two corresponding summands will cancel each other out. On the other hand, those partitions with no α -removable do not cancel out.

We conclude this argument using Lemma 1.1 we it was showed that, after rectification, those partitions (the ones without α -removable corners) are precisely the Schur functions appearing in $\Gamma_{(t|X)} s_\alpha$. \square

We finish this note with a useful form of this identity, see [12, 14, 2]. Let U_α be the operator multiplication by s_α , and let D_β be the skewing operator. That is, these adjoint operators are defined on the Schur basis by $U_\alpha(s_\lambda) = s_\lambda s_\alpha$, and $D_\beta(s_\lambda) = s_{\lambda/\beta}$. From Pieri's rule, and its dual, it follows that the vertex operator,

evaluated at $t = 1$, can be rewritten in the following elegant and well-known base-free way:

$$(1.5) \quad \Gamma_1 = \left(\sum_{i=0}^{\infty} U_{(i)} \right) \left(\sum_{j=0}^{\infty} (-1)^j D_{(1^j)} \right)$$

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